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ON THE CONJUGATION OF TRICOMPLEX NUMBERS: ALGEBRAIC OPERATIONS, MODULI, AND DECOMPOSITIONS

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ABSTRACT
We investigate the algebraic structure and symmetry properties of tricomplex numbers, emphasizing the seven distinct involutive conjugations arising from its independent imaginary units. After defining these conjugations, we derive their fundamental algebraic properties, including relations under addition, subtraction, and moduli behavior. By systematically comparing pairs of conjugations, we establish canonical identities that characterize elements of $\mathbb{C}_3$ in terms of invariant subalgebras and twisted modules. A total of 141 identities is obtained through analysis of equality, negation, addition, subtraction, and multiplicative relations among conjugation operators; among these are 112 canonical conjugation identities. These identities correspond to idempotent projection operators that decompose any tricomplex number into invariant and anti-invariant components, yielding a full canonical decomposition into symmetry-adapted subspaces. We also examine the action of conjugations on primitive idempotent elements. The results extend classical conjugation theory from complex analysis to higher-order multicomplex systems, providing a unified framework for coordinate-wise analysis, subalgebra classification, and functional formulations.

KEYWORDS: *Tricomplex Numbers; Multicomplex Algebra; Involutive Conjugations; Moduli; Idempotent Decomposition; Invariant Subalgebras; Primitive Idempotents.*
AMS Subject Classification: *30G35, 08A30, 13Jxx, 12Fxx.*

1. PRELIMINARIES AND NOTATION

Throughout this paper, we make systematic use of several distinguished subalgebras of the tricomplex algebra

$$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$$

generated by the commuting imaginary units i_1, i_2, i_3 with $i_k^2 = -1$.

(i) The real subalgebra is denoted by

$$\mathbb{C}_0 = \mathbb{R}$$

(ii) The **two–dimensional complex-type subalgebras** are

$$\mathbb{C}(i_k) = \{u + i_k v : u, v \in \mathbb{C}_0\}, \qquad k = 1,23,$$

together with

$$\mathbb{C}(i_1 i_2 i_3) = \{u + i_1 i_2 i_3 v : u, v \in \mathbb{C}_0\}$$

(iii) The **two–dimensional hyperbolic-type subalgebras** are

$$\mathbb{H}(i_j i_k) = \{u + i_j i_k v : u, v \in \mathbb{C}_0\}, \qquad 1 \le j < k \le 3.$$

(iv) The **four–dimensional bicomplex-type subalgebras** are

$$\mathbb{C}(i_1, i_2), \qquad \mathbb{C}(i_1, i_3), \qquad \mathbb{C}(i_2, i_3),$$

whose elements admit the canonical representation

$$x_1 + i_a x_2 + i_b x_3 + i_a i_b x_4, \qquad x_l \in \mathbb{C}_0$$

(v) The **four–dimensional mixed-type subalgebras**, are

$$\mathbb{C}(i_1, i_2 i_3), \qquad \mathbb{C}(i_2, i_1 i_2), \qquad \mathbb{C}(i_3, i_1 i_2),$$

with generic elements of the form

$$x_1 + i_a x_2 + i_b i_c x_3 + i_1 i_2 i_3 x_4, \qquad x_l \in \mathbb{C}_0.$$

(vi) Four–dimensional hyperbolic-pair subalgebra

$$\mathbb{H}(i_1 i_2, i_1 i_3) = \{x_1 + i_1 i_2 x_2 + i_1 i_3 x_3 + i_2 i_3 x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}.$$

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These subalgebras form the algebraic framework used throughout the paper for defining conjugations, moduli, and structural properties of tricomplex numbers.

2. INTRODUCTION TO CONJUGATION

Conjugation operators are fundamental in the study of tricomplex numbers, as they reflect the intrinsic symmetries of the algebra $\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$. Each conjugation is obtained by reversing the sign of one or more imaginary units and extends uniquely to an involutive automorphism of $\mathbb{C}_3$. These operators play a central role in defining moduli, invariants, and geometric properties of tricomplex numbers.

2.1 Conjugations of Tricomplex Numbers

In the tricomplex algebra $\mathbb{C}_3$, an element $\zeta \in \mathbb{C}_3$ can be expressed in the form

$$\zeta = \xi + i_3\eta \in \mathbb{C}_3, \qquad \xi, \eta \in \mathbb{C}(i_1, i_2),$$

and i_1, i_2, i_3 are independent imaginary units extending the classical complex number structure.

A conjugation is defined on ζ by reversing the sign of specified imaginary units, generalizing the complex conjugate to higher dimensions.

There are seven distinct involutive conjugation operators in $\mathbb{C}_3$, corresponding to the imaginary units and their nontrivial products. They are defined as follows:

(C1) i_1 − Conjugation

$$\overline{(\zeta)}_{i_1} = \overline{(\xi + i_3\,\eta)}_{i_1} = \overline{(\xi)}_{i_1} + i_3\,\overline{(\eta)}_{i_1}$$

(C2) i_2 − Conjugation

$$\overline{(\zeta)}_{i_2} = \overline{(\xi + i_3\eta)}_{i_2} = \overline{(\xi)}_{i_2} + i_3\,\overline{(\eta)}_{i_2}$$

(C3) i_3 − Conjugation

$$\overline{(\zeta)}_{i_3} = \overline{(\xi + i_3\eta)}_{i_3} = \overline{(\xi)}_{i_3} - i_3\,\overline{(\eta)}_{i_3} = \xi - i_3\,\eta$$

(C4) $i_1 i_2$ − Conjugation

$$\overline{(\zeta)}_{i_1 i_2} = \overline{(\xi + i_3\eta)}_{i_1 i_2} = \overline{(\xi)}_{i_1 i_2} + i_3\,\overline{(\eta)}_{i_1 i_2}$$

(C5) $i_1 i_3$ − Conjugation

$$\overline{(\zeta)}_{i_1 i_3} = \overline{(\xi + i_3\eta)}_{i_1 i_3} = \overline{(\xi)}_{i_1 i_3} - i_3\,\overline{(\eta)}_{i_1 i_3} = \overline{(\xi)}_{i_1} - i_3\,\overline{(\eta)}_{i_1}$$

(C6) $i_2 i_3$ − Conjugation

$$\overline{(\zeta)}_{i_2 i_3} = \overline{(\xi + i_3\eta)}_{i_2 i_3} = \overline{(\xi)}_{i_2 i_3} - i_3\,\overline{(\eta)}_{i_2 i_3} = \overline{(\xi)}_{i_2} - i_3\,\overline{(\eta)}_{i_2}$$

(C7) $i_1 i_2 i_3$ − Conjugation

$$\overline{(\zeta)}_{i_1 i_2 i_3} = \overline{(\xi + i_3\eta)}_{i_1 i_2 i_3} = \overline{(\xi)}_{i_1 i_2 i_3} - i_3\,\overline{(\eta)}_{i_1 i_2 i_3} = \overline{(\xi)}_{i_1 i_2} - i_3\,\overline{(\eta)}_{i_1 i_2}$$

2.2 Basic Conjugations Operators

To describe these operations more systematically, we introduce the notation

$$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{12}, \Gamma_{13}, \Gamma_{23}, \Gamma_{123},$$

each representing the corresponding conjugation operator that reverses the sign of specific imaginary units. For a general tricomplex number

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \in \mathbb{C}_3,$$

with coefficients $x_k \in \mathbb{C}_0$, the action of these operators is given by

$$\Gamma_1(\zeta) = x_1 - i_1x_2 + i_2x_3 + i_3x_4 - i_1i_2x_5 - i_1i_3x_6 + i_2i_3x_7 - i_1i_2i_3x_8,$$

$$\Gamma_2(\zeta) = x_1 + i_1x_2 - i_2x_3 + i_3x_4 - i_1i_2x_5 + i_1i_3x_6 - i_2i_3x_7 - i_1i_2i_3x_8,$$

$$\Gamma_3(\zeta) = x_1 + i_1x_2 + i_2x_3 - i_3x_4 + i_1i_2x_5 - i_1i_3x_6 - i_2i_3x_7 - i_1i_2i_3x_8,$$

$$\Gamma_{12}(\zeta) = x_1 - i_1x_2 - i_2x_3 + i_3x_4 + i_1i_2x_5 - i_1i_3x_6 - i_2i_3x_7 + i_1i_2i_3x_8,$$

$$\Gamma_{13}(\zeta) = x_1 - i_1x_2 + i_2x_3 - i_3x_4 - i_1i_2x_5 + i_1i_3x_6 - i_2i_3x_7 + i_1i_2i_3x_8,$$
$$\Gamma_{23}(\zeta) = x_1 + i_1x_2 - i_2x_3 - i_3x_4 - i_1i_2x_5 - i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8,$$
$$\Gamma_{123}(\zeta) = x_1 - i_1x_2 - i_2x_3 - i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 - i_1i_2i_3x_8.$$

2.3 Conjugations Expressed via the Decomposition $\xi + i_3\eta$

Using the decomposition $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi,\eta \in \mathbb{C}(i_1,i_2)$, the conjugation operators act as follows:

$$\Gamma_1(\zeta) = \Gamma_1(\xi) + i_3\Gamma_1(\eta),$$
$$\Gamma_2(\zeta) = \Gamma_2(\xi) + i_3\Gamma_2(\eta),$$
$$\Gamma_3(\zeta) = \Gamma_3(\xi) - i_3\Gamma_3(\eta) = \xi - i_3\,\eta,$$
$$\Gamma_{12}(\zeta) = \Gamma_{12}(\xi) + i_3\Gamma_{12}(\eta),$$
$$\Gamma_{13}(\zeta) = \Gamma_{13}(\xi) - i_3\Gamma_{13}(\eta) = \Gamma_1(\xi) - i_3\Gamma_1(\eta),$$
$$\Gamma_{23}(\zeta) = \Gamma_{23}(\xi) - i_3\Gamma_{23}(\eta) = \Gamma_2(\xi) - i_3\Gamma_2(\eta),$$
$$\Gamma_{123}(\zeta) = \Gamma_{123}(\xi) - i_3\Gamma_{123}(\eta) = \Gamma_{12}(\xi) - i_3\Gamma_{12}(\eta).$$

3. CONJUGATION-BASED PROPOSITIONS IN TRICOMPLEX ALGEBRA

In the tricomplex algebra $\mathbb{C}_3 = \mathbb{C}(i_1,i_2,i_3)$, the seven involutive conjugations introduced in Section 2 encode the algebra's inherent symmetries. Each conjugation operator

$$\Gamma_u: \mathbb{C}_3 \to \mathbb{C}_3, \qquad u \in \{1,2,3,12,13,23,123\},$$

is an involutive automorphism of $\mathbb{C}_3$, meaning that for all $\zeta,\eta \in \mathbb{C}_3$:

$$\Gamma_u(\Gamma_u(\zeta)) = \zeta, \qquad \Gamma_u(\zeta+\eta) = \Gamma_u(\zeta) + \Gamma_u(\eta), \qquad \Gamma_u(\zeta\eta) = \Gamma_u(\zeta)\Gamma_u(\eta).$$

The fundamental role of these conjugations is that they identify invariant subspaces and subalgebras of $\mathbb{C}_3$, and thus characterize both algebraic and geometric properties of tricomplex elements.

3.1 Classification by Conjugation Relations

We now develop a family of propositions that classify elements of $\mathbb{C}_3$ based on relations between pairs of distinct conjugation operators. For a fixed decomposition

$$\zeta = \xi + i_3\eta \in \mathbb{C}_3, \qquad \xi,\eta \in \mathbb{C}(i_1,i_2),$$

the pattern of equalities or sign-negations between two conjugations determines precisely which subalgebra or twisted component of $\mathbb{C}_3$ the element ζ belongs to.

Let Γ_u and Γ_v be any two distinct conjugation operators from the set

$$\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{12}, \Gamma_{13}, \Gamma_{23}, \Gamma_{123}\}.$$

Then for any fixed $\zeta \in \mathbb{C}_3$, exactly one of the following relations holds:

1. Equality Relation

$$\Gamma_u(\zeta) = \Gamma_v(\zeta),$$

which imposes a structural constraint on ζ and forces it to lie in a specific invariant subalgebra of $\mathbb{C}_3$.

2. Negation Relation

$$\Gamma_u(\zeta) = -\Gamma_v(\zeta),$$

which restricts ζ to a twisted subspace of $\mathbb{C}_3$ formed by left multiplication of a standard subalgebra by a designated imaginary unit. The set of all such relations between pairs of conjugations gives rise to a family of propositions - specifically, seven principal propositions with a total of 56 distinct results.

Proposition 3.1 (Fixed and Anti-fixed Sets of Conjugations in $\mathbb{C}_3$)

(A) Fixed points

For $\zeta \in \mathbb{C}_3$, the following equivalences hold:

(P1) $\Gamma_1(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_3)$

(P2) $\Gamma_2(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_3)$

(P3) $\Gamma_3(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2)$

(P4) $\Gamma_{12}(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_3, i_1 i_2)$

(P5) $\Gamma_{13}(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_1 i_3)$

(P6) $\Gamma_{23}(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2 i_3)$

(P7) $\Gamma_{123}(\zeta) = \zeta \Leftrightarrow \zeta \in \mathbb{H}(i_1 i_2, i_1 i_3)$

(B) Anti-fixed points

(P8) $\Gamma_1(\zeta) = -\zeta \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_3)$

(P9) $\Gamma_2(\zeta) = -\zeta \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_3)$

(P10) $\Gamma_3(\zeta) = -\zeta \Leftrightarrow \zeta \in i_3\mathbb{C}(i_1, i_2)$

(P11) $\Gamma_{12}(\zeta) = -\zeta \Leftrightarrow \zeta \in i_1\mathbb{C}(i_3, i_1 i_2)$

(P12) $\Gamma_{13}(\zeta) = -\zeta \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_1 i_3)$

(P13) $\Gamma_{23}(\zeta) = -\zeta \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_2 i_3)$

(P14) $\Gamma_{123}(\zeta) = -\zeta \Leftrightarrow \zeta \in i_1\mathbb{H}(i_1 i_2, i_1 i_3)$

Proof (Representative case):

We prove only **(P1)**; all other cases follow analogously.

Let

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8 \in \mathbb{C}_3.$$

By definition

$$\Gamma_1(\zeta) = x_1 - i_1 x_2 + i_2 x_3 + i_3 x_4 - i_1 i_2 x_5 - i_1 i_3 x_6 + i_2 i_3 x_7 - i_1 i_2 i_3 x_8.$$

Thus $\Gamma_1(\zeta) = \zeta$ if and only if

$$x_2 = x_5 = x_6 = x_8 = 0,$$

which gives

$$\zeta = x_1 + i_2 x_3 + i_3 x_4 + i_2 i_3 x_7 \in \mathbb{C}(i_2, i_3).$$

Conversely, every element of $\mathbb{C}(i_2, i_3)$ is unchanged by Γ_1.

Hence **(P1)** holds.

□

Proposition 3.2 (Equality and Negation of Tricomplex Conjugations in $\mathbb{C}_3$)

Let $\zeta \in \mathbb{C}_3$, then the coincidence and negation of tricomplex conjugations with respect to i_1 completely characterize the underlying subalgebra structure of ζ.

(A) Equality of Conjugations

The equality of Γ_1 with other conjugations holds if and only if ζ belongs to the corresponding canonical subalgebra:

(P15) $\Gamma_1(\zeta) = \Gamma_2(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_3, i_1 i_2)$

(P16) $\Gamma_1(\zeta) = \Gamma_3(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_1 i_3)$

(P17) $\Gamma_1(\zeta) = \Gamma_{12}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_3)$

(P18) $\Gamma_1(\zeta) = \Gamma_{13}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2)$

(P19) $\Gamma_1(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{H}(i_1 i_2, i_1 i_3)$

(P20) $\Gamma_1(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2 i_3)$

(B) Negated Conjugations

Similarly, the negation of conjugations with respect to Γ_1 determines "twisted" subalgebras generated by multiplication with imaginary units:

(P21) $\Gamma_1(\zeta) = -\Gamma_2(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_3, i_1 i_2)$

(P22) $\Gamma_1(\zeta) = -\Gamma_3(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_1 i_3)$

(P23) $\Gamma_1(\zeta) = -\Gamma_{12}(\zeta) \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_3)$

(P24) $\Gamma_1(\zeta) = -\Gamma_{13}(\zeta) \Leftrightarrow \zeta \in i_3\mathbb{C}(i_1, i_2)$

(P25) $\Gamma_1(\zeta) = -\Gamma_{23}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{H}(i_1 i_2, i_1 i_3)$

(P26) $\Gamma_1(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_2 i_3)$

Interpretation

Equality of conjugations identifies invariant subalgebras of $\mathbb{C}_3$, whereas negated conjugations describe twisted subalgebras obtained via multiplication by imaginary units. Together, these relations provide a complete structural classification of elements in $\mathbb{C}_3$ based on conjugation symmetries.

Proof of (P15) :

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_1(\zeta) = \Gamma_1(\xi) + i_3\Gamma_1(\eta), \qquad \Gamma_2(\zeta) = \Gamma_2(\xi) + i_3\Gamma_2(\eta).$$

Then

$$\Gamma_1(\zeta) = \Gamma_2(\zeta) \Leftrightarrow \Gamma_1(\xi) = \Gamma_2(\xi) \text{ and } \Gamma_1(\eta) = \Gamma_2(\eta)$$

This implies

$$\xi = u + i_1 i_2 v, \qquad \eta = \mu + i_1 i_2 \vartheta, \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_1 i_2 v + i_3(\mu + i_1 i_2 \vartheta) = u + i_3\mu + i_1 i_2 v + i_1 i_2 i_3 \vartheta \in \mathbb{C}(i_3, i_1 i_2).$$

Conversely, if $\zeta \in \mathbb{C}(i_3, i_1 i_2)$, then its decomposition immediately yields

$$\Gamma_1(\zeta) = \Gamma_2(\zeta).$$

□

All other cases **(P16-26)** follow analogously.

Proposition 3.3 (Equality and Negation of Tricomplex Conjugations under i_2)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3$ with $\xi, \eta \in \mathbb{C}(i_1, i_2)$.

Then the coincidence and negation of tricomplex conjugations with respect to i_2 completely determine the subalgebra structure of ζ.

(A) Equality of Conjugations

The equality of Γ_2 with other conjugations holds if and only if ζ belongs to the corresponding canonical subalgebra:

(P27) $\Gamma_2(\zeta) = \Gamma_3(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2 i_3)$

(P28) $\Gamma_2(\zeta) = \Gamma_{12}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_3)$

(P29) $\Gamma_2(\zeta) = \Gamma_{13}(\zeta) \Leftrightarrow \zeta \in \mathbb{H}(i_1 i_2, i_1 i_3)$

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(P30) $\Gamma_2(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2)$

(P31) $\Gamma_2(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_1 i_3)$

(B) Negated Conjugations

Similarly, negation of conjugations with respect to Γ_2 identifies twisted subalgebras:

(P32) $\Gamma_2(\zeta) = -\Gamma_3(\zeta) \Leftrightarrow \zeta \in i_2\,\mathbb{C}(i_1, i_2 i_3)$,

(P33) $\Gamma_2(\zeta) = -\Gamma_{12}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_3)$,

(P34) $\Gamma_2(\zeta) = -\Gamma_{13}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{H}(i_1 i_2, i_1 i_3)$,

(P35) $\Gamma_2(\zeta) = -\Gamma_{23}(\zeta) \Leftrightarrow \zeta \in i_3\mathbb{C}(i_1, i_2)$,

(P36) $\Gamma_2(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_1 i_3)$.

Interpretation:

Equality of conjugations identifies invariant subalgebras of $\mathbb{C}_3$, while negated conjugations describe twisted subalgebras generated via multiplication by imaginary units i_1 or i_2. Together, these provide a complete structural classification of elements in $\mathbb{C}_3$ with respect to i_2-conjugation.

Proof of (P27) (Explicit):

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_2(\zeta) = \Gamma_2(\xi) + i_3\Gamma_2(\eta), \qquad \Gamma_3(\zeta) = \xi - i_3\eta.$$

Then

$$\Gamma_2(\zeta) = \Gamma_3(\zeta) \Leftrightarrow \Gamma_2(\xi) = \xi \ \text{ and } \ \Gamma_2(\eta) = -\eta.$$

This implies

$$\xi = u + i_1 v\,, \qquad \eta = i_2(\mu + i_1\vartheta), \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_1 v + i_3(i_2\mu + i_1 i_2\vartheta) = u + i_1 v + i_2 i_3\mu + +i_1 i_2 i_3\vartheta \in \mathbb{C}(i_1, i_2 i_3).$$

Conversely, if $\zeta \in \mathbb{C}(i_1, i_2 i_3)$, then the above decomposition immediately yields

$$\Gamma_2(\zeta) = \Gamma_3(\zeta).$$

This completes the proof.

□

All other cases **(P28-36)** follow analogously.

Proposition 3.4 (Equality and Negation of i_3 −Conjugations in $\mathbb{C}_3$)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3$ with $\xi, \eta \in \mathbb{C}(i_1, i_2)$. Then the coincidence or negation of the i_3 −conjugation with other tricomplex conjugations uniquely determines the subalgebra of $\mathbb{C}_3$ to which ζ belongs.

(A) Equality of Conjugations

(P37) $\Gamma_3(\zeta) = \Gamma_{12}(\zeta) \Leftrightarrow \zeta \in \mathbb{H}(i_1 i_2, i_1 i_3)$

(P38) $\Gamma_3(\zeta) = \Gamma_{13}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_3)$

(P39) $\Gamma_3(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_3)$

(P40) $\Gamma_3(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_3, i_1 i_2)$

(B) Negated i_3 −Conjugation: Twisted Subalgebra Characterization

(P41) $\Gamma_3(\zeta) = -\Gamma_{12}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{H}(i_1 i_2, i_1 i_3)$

(P42) $\Gamma_3(\zeta) = -\Gamma_{13}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_3)$

(P43) $\Gamma_3(\zeta) = -\Gamma_{23}(\zeta) \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_3)$

(P44) $\Gamma_3(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_3, i_1 i_2)$

Interpretation

Equality of i_3- conjugation with another conjugation identifies invariant two-generator subalgebras of $\mathbb{C}_3$. Negation of i_3 −conjugation produces twisted subalgebras obtained by multiplication with i_1 or i_2. Together, these relations yield a complete structural classification of elements in $\mathbb{C}_3$ based on i_3 −conjugation symmetries.

Proof (Representative Case: P37)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_3(\zeta) = \xi - i_3\eta, \qquad \Gamma_{12}(\zeta) = \Gamma_{12}(\xi) + i_3\Gamma_{12}(\eta).$$

Then

$$\Gamma_3(\zeta) = \Gamma_{12}(\zeta) \Leftrightarrow \xi = \Gamma_{12}(\xi) \text{ and } \eta = -\Gamma_{12}(\eta).$$

This implies

$$\xi = u + i_1 i_2 v, \qquad \eta = i_1\mu + i_2\vartheta, \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_1 i_2 v + i_3(i_1\mu + i_2\vartheta) = u + i_1 i_2 v + i_1 i_3\mu + i_2 i_3\vartheta \in \mathbb{H}(i_1 i_2, i_1 i_3).$$

Conversely, if $\zeta \in \mathbb{H}(i_1 i_2, i_1 i_3)$, then the above decomposition immediately yields

$$\Gamma_3(\zeta) = \Gamma_{12}(\zeta).$$

This completes the proof.

□

All remaining statements follow by exactly the same argument, using the corresponding conjugation decomposition formulas. Hence, their proofs are omitted as they are entirely analogous.

Proposition 3.5 ($i_1 i_2$ −Conjugation and Subalgebra Characterization in $\mathbb{C}_3$)

Let

$$\zeta = \xi + i_3\eta \in \mathbb{C}_3 \text{ with } \xi, \eta \in \mathbb{C}(i_1, i_2).$$

Then the relationship between the $i_1 i_2$ −conjugation $\Gamma_{12}(\zeta)$ and the other conjugations completely characterizes the subalgebra of $\mathbb{C}_3$ to which ζ belongs.

(A) Equality Case: Standard Subalgebras

If the $i_1 i_2$ −conjugation coincides with another conjugation, then ζ lies in a two-generator (or mixed) subalgebra of $\mathbb{C}_3$:

(P45) $\Gamma_{12}(\zeta) = \Gamma_{13}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2 i_3)$

(P46) $\Gamma_{12}(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_1 i_3)$

(P47) $\Gamma_{12}(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_2)$

(B) Negation Case: Twisted Subalgebras

If the $i_1 i_2$ −conjugation equals the **negative** of another conjugation, then ζ belongs to a twisted subalgebra obtained by multiplication with a basis element i_1, i_2, or i_3 :

(P48) $\Gamma_{12}(\zeta) = -\Gamma_{13}(\zeta) \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_2 i_3)$

(P49) $\Gamma_{12}(\zeta) = -\Gamma_{23}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_1 i_3)$

(P50) $\Gamma_{12}(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_3\mathbb{C}(i_1, i_2)$

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Interpretation

The behavior of the $i_1 i_2$ −conjugation relative to other conjugations provides a complete classification of subalgebra membership in $\mathbb{C}_3$:

- **Equality of conjugations** constrains ζ to lie in a standard two-generator subalgebra**.**
- **Negation of conjugations** forces ζ into a twisted subalgebra, obtained by multiplication with one of the generators i_1, i_2, or i_3.

Thus, conjugation symmetry and sign reversal together encode both the structural and twisted components of $\mathbb{C}_3$.

Proof of (P45)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_{12}(\zeta) = \Gamma_{12}(\xi) + i_3\Gamma_{12}(\eta), \qquad \Gamma_{13}(\zeta) = \Gamma_1(\xi) - i_3\Gamma_1(\eta).$$

Then

$$\Gamma_{12}(\zeta) = \Gamma_{13}(\zeta) \Leftrightarrow \Gamma_{12}(\xi) = \Gamma_1(\xi) \text{ and } \Gamma_{12}(\eta) = -\Gamma_1(\eta).$$

This implies

$$\xi = u + i_1 v, \qquad \eta = i_2\mu + i_1 i_2\vartheta, \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_1 v + i_3(i_2\mu + i_1 i_2\vartheta) = u + i_1 v + i_2 i_3\mu + +i_1 i_2 i_3\vartheta \in \mathbb{C}(i_1, i_2 i_3).$$

Conversely, if $\zeta \in \mathbb{C}(i_1, i_2 i_3)$, then the above decomposition immediately yields

$$\Gamma_{12}(\zeta) = \Gamma_{13}(\zeta).$$

□

All other cases **(P46-50)** follow analogously.

Proposition 3.6 (Unified Conjugation-Based Subalgebra Characterization for $i_1 i_3$)

Let

$$\zeta = \xi + i_3\eta \in \mathbb{C}_3 \text{ with } \xi, \eta \in \mathbb{C}(i_1, i_2).$$

Then the relation between the $i_1 i_3$ −conjugation $\Gamma_{13}(\zeta)$ and the other conjugations Γ_{23} and Γ_{123} determines the subalgebra containing ζ in one of the following six mutually exclusive ways:

(A) Standard Subalgebra Membership (Equality Case)

(P51) $\Gamma_{13}(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_3, i_1 i_2)$

(P52) $\Gamma_{13}(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_1, i_3)$

(B) Twisted Subalgebra Membership (Negation Case)

(P53) $\Gamma_{13}(\zeta) = -\Gamma_{23}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_3, i_1 i_2)$

(P54) $\Gamma_{13}(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_2\mathbb{C}(i_1, i_3)$

Interpretation

- Equality of conjugations places ζ in a conventional subalgebra generated by the indicated basis elements.
- Negation of conjugations places ζ in a twisted component, which is the product of a basis unit with a conventional two-generator subalgebra.

Proof (Case: P51)

$\Gamma_{13}(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_3, i_1 i_2)$

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3,\ \xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_{13}(\zeta) = \Gamma_1(\xi) - i_3\Gamma_1(\eta), \qquad \Gamma_{23}(\zeta) = \Gamma_2(\xi) - i_3\Gamma_2(\eta).$$

Then

$$\Gamma_{13}(\zeta) = \Gamma_{23}(\zeta) \Leftrightarrow \Gamma_1(\xi) = \Gamma_2(\xi) \text{ and } \Gamma_1(\eta) = \Gamma_2(\eta).$$

This implies

$$\xi = u + i_1 i_2 v, \qquad \eta = u + i_1 i_2 v, \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_1 i_2 v + i_3(u + i_1 i_2 v) = u + i_3\mu + i_1 i_2 v + i_1 i_2 i_3 \vartheta \in \mathbb{C}(i_3, i_1 i_2).$$

Conversely, if $\zeta \in \mathbb{C}(i_3, i_1 i_2)$, then the above decomposition immediately yields

$$\Gamma_{13}(\zeta) = \Gamma_{23}(\zeta).$$

□

Analogously, we prove **(P52-54)**

Proposition 3.7 (Equality and Negation of $i_2 i_3$ − Conjugation)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3$ with $\xi, \eta \in \mathbb{C}(i_1, i_2)$.Then:

(P55) $\Gamma_{23}(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \zeta \in \mathbb{C}(i_2, i_3)$

(P56) $\Gamma_{23}(\zeta) = -\Gamma_{123}(\zeta) \Leftrightarrow \zeta \in i_1\mathbb{C}(i_2, i_3)$

Proof (Case: P55)

Let $\zeta = \xi + i_3\eta \in \mathbb{C}_3$, $\xi, \eta \in \mathbb{C}(i_1, i_2)$. Using the decomposition formulas:

$$\Gamma_{23}(\zeta) = \Gamma_2(\xi) - i_3\Gamma_2(\eta), \qquad \Gamma_{123}(\zeta) = \Gamma_{12}(\xi) - i_3\Gamma_{12}(\eta).$$

Then

$$\Gamma_{23}(\zeta) = \Gamma_{123}(\zeta) \Leftrightarrow \Gamma_2(\xi) = \Gamma_{12}(\xi) \text{ and } \Gamma_2(\eta) = \Gamma_{12}(\eta).$$

This implies

$$\xi = u + i_2 v, \qquad \eta = \mu + i_2\vartheta, \qquad u, v, \mu, \vartheta \in \mathbb{C}_0$$

Hence,

$$\zeta = \xi + i_3\eta = u + i_2 v + i_3(\mu + i_2\vartheta) = u + i_2 v + i_3\mu + i_2 i_3\vartheta \in \mathbb{C}(i_2, i_3).$$

Conversely, if $\zeta \in \mathbb{C}(i_2, i_3)$, then the above decomposition immediately yields

$$\Gamma_{23}(\zeta) = \Gamma_{123}(\zeta).$$

This completes the proof.

□

Analogously, we prove **(P56)**

4. Linear Combinations of Conjugations in Tricomplex Algebra $\mathbb{C}_3$:

The following identities describe sums and differences of conjugations, which project ζ onto canonical bicomplex subspaces or their imaginary-unit scaled modules.

I. Sums with the original number

(P57) $\zeta + \Gamma_1(\zeta) = 2(x_1 + i_2 x_3 + i_3 x_4 + i_2 i_3 x_7) \in \mathbb{C}(i_2, i_3)$

(P58) $\zeta + \Gamma_2(\zeta) = 2(x_1 + i_1 x_2 + i_3 x_4 + i_1 i_3 x_6) \in \mathbb{C}(i_1, i_3)$

(P59) $\zeta + \Gamma_3(\zeta) = 2(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_5) \in \mathbb{C}(i_1, i_2)$

(P60) $\zeta + \Gamma_{12}(\zeta) = 2(x_1 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_2 i_3 x_8) \in \mathbb{C}(i_3, i_1 i_2)$

(P61) $\zeta + \Gamma_{13}(\zeta) = 2(x_1 + i_2 x_3 + i_1 i_3 x_6 + i_1 i_2 i_3 x_8) \in \mathbb{C}(i_2, i_1 i_3)$

(P62) $\zeta + \Gamma_{23}(\zeta) = 2(x_1 + i_1x_2 + i_2i_3x_7 + i_1i_2i_3x_8) \in \mathbb{C}(i_1, i_2i_3)$

(P63) $\zeta + \Gamma_{123}(\zeta) = 2(x_1 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7) \in \mathbb{C}(i_1i_2, i_1i_3)$

II. Differences with the original number

(P64) $\zeta - \Gamma_1(\zeta) = 2i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8) \in i_1\mathbb{C}(i_2, i_3)$

(P65) $\zeta - \Gamma_2(\zeta) = 2i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8) \in i_2\mathbb{C}(i_1, i_3)$

(P66) $\zeta - \Gamma_3(\zeta) = 2i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8) \in i_3\mathbb{C}(i_1, i_2)$

(P67) $\zeta - \Gamma_{12}(\zeta) = 2i_1(x_2 + i_3x_6 - i_1i_2x_3 - i_1i_2i_3x_7) \in i_1\mathbb{C}(i_3, i_1i_2)$

(P68) $\zeta - \Gamma_{13}(\zeta) = 2i_1(x_2 + i_2x_5 - i_1i_3x_4 - i_1i_2i_3x_7) \in i_1\mathbb{C}(i_2, i_1i_3)$

(P69) $\zeta - \Gamma_{23}(\zeta) = 2i_2(x_3 + i_1x_5 - i_2i_3x_4 - i_1i_2i_3x_6) \in i_2\mathbb{C}(i_1, i_2i_3)$

(P70) $\zeta - \Gamma_{123}(\zeta) = 2i_1(x_2 - i_1i_2x_3 - i_1i_3x_4 + i_2i_3x_8) \in i_1\mathbb{C}(i_1i_2, i_1i_3)$

III. Pairwise sums and differences of conjugations

(P71) $\Gamma_1(\zeta) + \Gamma_2(\zeta) = 2(x_1 + i_3x_4 - i_1i_2x_5 - i_1i_2i_3x_8) \in \mathbb{C}(i_3, i_1i_2)$

(P72) $\Gamma_1(\zeta) + \Gamma_3(\zeta) = 2(x_1 + i_2x_3 - i_1i_3x_6 - i_1i_2i_3x_8) \in \mathbb{C}(i_2, i_1i_3)$

(P73) $\Gamma_1(\zeta) + \Gamma_{12}(\zeta) = 2(x_1 - i_1x_2 + i_3x_4 - i_1i_3x_6) \in \mathbb{C}(i_1, i_3)$

(P74) $\Gamma_1(\zeta) + \Gamma_{13}(\zeta) = 2(x_1 - i_1x_2 + i_2x_3 - i_1i_2x_5) \in \mathbb{C}(i_1, i_2)$

(P75) $\Gamma_1(\zeta) + \Gamma_{23}(\zeta) = 2(x_1 - i_1i_2x_5 - i_1i_3x_6 + i_2i_3x_7) \in \mathbb{C}(i_1i_2, i_1i_3)$

(P76) $\Gamma_1(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_1x_2 + i_2i_3x_7 - i_1i_2i_3x_8) \in \mathbb{C}(i_1, i_2i_3)$

(P77) $\Gamma_1(\zeta) - \Gamma_2(\zeta) = 2i_1(-x_2 - i_3x_6 - i_1i_2x_3 - i_1i_2i_3x_7) \in \mathbb{C}(i_3, i_1i_2)$

(P78) $\Gamma_1(\zeta) - \Gamma_3(\zeta) = 2i_1(-x_2 - i_2x_5 - i_1i_3x_4 - i_1i_2i_3x_7) \in \mathbb{C}(i_2, i_1i_3)$

(P79) $\Gamma_1(\zeta) - \Gamma_{12}(\zeta) = 2i_2(x_3 - i_1x_5 + i_3x_7 - i_1i_3x_8) \in \mathbb{C}(i_1, i_3)$

(P80) $\Gamma_1(\zeta) - \Gamma_{13}(\zeta) = 2i_3(\mathrm{x}_4 - \mathrm{i}_1\mathrm{x}_6 + \mathrm{i}_2\mathrm{x}_7 - \mathrm{i}_1\mathrm{i}_2\mathrm{x}_8) \in \mathbb{C}(i_1, i_2)$

(P81) $\Gamma_1(\zeta) - \Gamma_{23}(\zeta) = 2(-i_1x_2 + i_2x_3 + i_3x_4 - i_1i_2i_3x_8) \in \mathbb{C}(i_1, i_2)$

(P82) $\Gamma_1(\zeta) - \Gamma_{123}(\zeta) = 2(i_2x_3 + i_3x_4 - i_1i_2x_5 - i_1i_3x_6)$

(P83) $\Gamma_2(\zeta) + \Gamma_3(\zeta) = 2(x_1 + i_1x_2 - i_2i_3x_7 - i_1i_2i_3x_8)$

(P84) $\Gamma_2(\zeta) + \Gamma_{12}(\zeta) = 2(x_1 - i_2x_3 + i_3x_4 - i_2i_3x_7)$

(P85) $\Gamma_2(\zeta) + \Gamma_{13}(\zeta) = 2(x_1 - i_1i_2x_5 + i_1i_3x_6 - i_2i_3x_7)$

(P86) $\Gamma_2(\zeta) + \Gamma_{23}(\zeta) = 2(x_1 + i_1x_2 - i_2x_3 - i_1i_2x_5)$

(P87) $\Gamma_2(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_2x_3 + i_1i_3x_6 - i_1i_2i_3x_8)$

(P88) $\Gamma_2(\zeta) - \Gamma_3(\zeta) = 2(-i_2x_3 + i_3x_4 - i_1i_2x_5 + i_1i_3x_6)$

(P89) $\Gamma_2(\zeta) - \Gamma_{12}(\zeta) = 2(i_1x_2 - i_1i_2x_5 + i_1i_3x_6 - i_1i_2i_3x_8)$

(P90) $\Gamma_2(\zeta) - \Gamma_{13}(\zeta) = 2(i_1x_2 - i_2x_3 + i_3x_4 - i_1i_2i_3x_8)$

(P91) $\Gamma_2(\zeta) - \Gamma_{23}(\zeta) = 2(i_3x_4 + i_1i_3x_6 - i_2i_3x_7 - i_1i_2i_3x_8)$

(P92) $\Gamma_2(\zeta) - \Gamma_{123}(\zeta) = 2(i_1x_2 + i_3x_4 - i_1i_2x_5 - i_2i_3x_7)$

(P93) $\Gamma_3(\zeta) + \Gamma_{12}(\zeta) = 2(x_1 + i_1i_2x_5 - i_1i_3x_6 - i_2i_3x_7)$

(P94) $\Gamma_3(\zeta) + \Gamma_{13}(\zeta) = 2(x_1 + i_2x_3 - i_3x_4 - i_2i_3x_7)$

(P95) $\Gamma_3(\zeta) + \Gamma_{23}(\zeta) = 2(x_1 + i_1x_2 - i_3x_4 - i_1i_3x_6)$

(P96) $\Gamma_3(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_3x_4 + i_1i_2x_5 - i_1i_2i_3x_8)$

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(P97) $\Gamma_3(\zeta) - \Gamma_{12}(\zeta) = 2(i_1x_2 + i_2x_3 - i_3x_4 - i_1i_2i_3x_8)$
(P98) $\Gamma_3(\zeta) - \Gamma_{13}(\zeta) = 2(i_1x_2 + i_1i_2x_5 - i_1i_3x_6 - i_1i_2i_3x_8)$
(P99) $\Gamma_3(\zeta) - \Gamma_{23}(\zeta) = 2(i_2x_3 + i_1i_2x_5 - i_2i_3x_7 - i_1i_2i_3x_8)$
(P100) $\Gamma_3(\zeta) - \Gamma_{123}(\zeta) = 2(i_1x_2 + i_2x_3 - i_1i_3x_6 - i_2i_3x_7)$
(P101) $\Gamma_{12}(\zeta) + \Gamma_{13}(\zeta) = 2(x_1 - i_1x_2 - i_2i_3x_7 + i_1i_2i_3x_8)$
(P102) $\Gamma_{12}(\zeta) + \Gamma_{23}(\zeta) = 2(x_1 - i_2x_3 - i_1i_3x_6 + i_1i_2i_3x_8)$
(P103) $\Gamma_{12}(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_1x_2 - i_2x_3 + i_1i_2x_5)$
(P104) $\Gamma_{12}(\zeta) - \Gamma_{13}(\zeta) = 2(-i_2x_3 + i_3x_4 + i_1i_2x_5 - i_1i_3x_6)$
(P105) $\Gamma_{12}(\zeta) - \Gamma_{23}(\zeta) = 2(-i_1x_2 + i_3x_4 + i_1i_2x_5 - i_2i_3x_7)$
(P106) $\Gamma_{12}(\zeta) - \Gamma_{123}(\zeta) = 2(i_3x_4 - i_1i_3x_6 - i_2i_3x_7 + i_1i_2i_3x_8)$
(P107) $\Gamma_{13}(\zeta) + \Gamma_{23}(\zeta) = 2(x_1 - i_3x_4 - i_1i_2x_5 + i_1i_2i_3x_8)$
(P108) $\Gamma_{13}(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_1x_2 - i_3x_4 + i_1i_3x_6)$
(P109) $\Gamma_{13}(\zeta) - \Gamma_{23}(\zeta) = 2(-i_1x_2 + i_2x_3 + i_1i_3x_6 - i_2i_3x_7)$
(P110) $\Gamma_{13}(\zeta) - \Gamma_{123}(\zeta) = 2(i_2x_3 - i_1i_2x_5 - i_2i_3x_7 + i_1i_2i_3x_8)$
(P111) $\Gamma_{23}(\zeta) + \Gamma_{123}(\zeta) = 2(x_1 - i_2x_3 - i_3x_4 + i_2i_3x_7)$
(P112) $\Gamma_{23}(\zeta) - \Gamma_{123}(\zeta) = 2(i_1x_2 - i_1i_2x_5 - i_1i_3x_6 + i_1i_2i_3x_8)$

4.1 Analysis of Identities (P57–P112)

Identities **(P57)–(P112)** arise from linear combinations of conjugations

$$\zeta \pm \Gamma_u(\zeta), \quad \Gamma_u(\zeta) \pm \Gamma_v(\zeta),$$

in the tricomplex algebra $\mathbb{C}_3$. They systematically project a general tricomplex number onto specific invariant subspaces or twisted modules — for example, bicomplex subalgebras

$$\mathbb{C}(i_j, i_k) \ or \ \ i_j\mathbb{C}(i_k, i_l).$$

These relations reflect the underlying abelian group of involutive conjugations: sums isolate components symmetric under given conjugations, and differences isolate antisymmetric components. Equivalently, the identities can be expressed using idempotent projectors

$$P_u^+(\zeta) = \frac{1}{2}(I + \Gamma_u), \quad P_u^- = \frac{1}{2}(I - \Gamma_u),$$

$$P_u^+(\zeta) = \frac{1}{2}\big(\zeta + \Gamma_u(\zeta)\big), \quad P_u^-(\zeta) = \frac{1}{2}\big(\zeta - \Gamma_u(\zeta)\big).$$

which decompose ζ into invariant and anti-invariant parts. This provides a complete canonical decomposition of $\mathbb{C}_3$, facilitating coordinate-wise analysis, invariant subspace classification, and functional/operational formulations in tricomplex and multicomplex theory.

5. MODULI OF TRICOMPLEX NUMBERS

In the tricomplex number system

$$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3),$$

several distinct moduli can be defined, depending on the choice of the underlying bicomplex subspace. Each modulus is constructed via suitable conjugation operators, and every bicomplex subsystem of $\mathbb{C}_3$ admits exactly four quadratic moduli.

In this section, all such moduli are presented in a systematic and unified manner, classified according to the corresponding bicomplex algebra.

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(M1) Moduli in $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2})$

The bicomplex algebra $\mathbb{C}(i_1, i_2)$ admits four distinct moduli, each generated by a compatible pair of conjugations.

(P113) $\zeta\ \Gamma_3(\zeta)$

(P114) $\Gamma_1(\zeta)\ \Gamma_{13}(\zeta)$

(P115) $\Gamma_2(\zeta)\ \Gamma_{23}(\zeta)$

(P116) $\Gamma_{12}(\zeta)\ \Gamma_{123}(\zeta)$

(M2) Moduli in $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_3})$

Similarly, the bicomplex subspace $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_3})$ possesses four admissible moduli given by

(P117) $\zeta\ \Gamma_2(\zeta)$

(P118) $\Gamma_1(\zeta)\ \Gamma_{12}(\zeta)$

(P119) $\Gamma_3(\zeta)\ \Gamma_{23}(\zeta)$

(P120) $\Gamma_{13}(\zeta)\ \Gamma_{123}(\zeta)$

(M3) Moduli in $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_3})$

The four moduli associated with the bicomplex algebra $\mathbb{C}(i_2, i_3)$ are

(P121) $\zeta\ \Gamma_1(\zeta)$

(P122) $\Gamma_2(\zeta)\ \Gamma_{12}(\zeta)$

(P123) $\Gamma_3(\zeta)\ \Gamma_{13}(\zeta)$

(P124) $\Gamma_{23}(\zeta)\ \Gamma_{123}(\zeta)$

(M4) Moduli in $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2 i_3})$

For the bicomplex structure generated by i_1 and $i_2 i_3$, the corresponding moduli are

(P125) $\zeta\ \Gamma_{23}(\zeta)$

(P126) $\Gamma_1(\zeta)\ \Gamma_{123}(\zeta)$

(P127) $\Gamma_2(\zeta)\ \Gamma_3(\zeta)$

(P128) $\Gamma_{12}(\zeta)\ \Gamma_{13}(\zeta)$

(M5) Moduli in $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_1 i_3})$

The bicomplex algebra $\mathbb{C}(i_2, i_1 i_3)$ admits the following four moduli:

(P129) $\zeta\ \Gamma_{13}(\zeta)$

(P130) $\Gamma_1(\zeta)\ \Gamma_3(\zeta)$

(P131) $\Gamma_2(\zeta)\ \Gamma_{123}(\zeta)$

(P132) $\Gamma_{12}(\zeta)\ \Gamma_{23}(\zeta)$

(M6) Moduli in $\mathbb{C}(\boldsymbol{i_3}, \boldsymbol{i_1 i_2})$

For the bicomplex subspace $\mathbb{C}(i_3, i_1 i_2)$, the associated moduli are

(P133) $\zeta\ \Gamma_{12}(\zeta)$

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(P134) $\Gamma_1(\zeta)\ \Gamma_2(\zeta)$

(P135) $\Gamma_3(\zeta)\ \Gamma_{123}(\zeta)$

(P136) $\Gamma_{13}(\zeta)\ \Gamma_{23}(\zeta)$

(M7) Moduli in $\mathbb{C}(i_1 i_2, i_1 i_3)$

Finally, the bicomplex algebra generated by $i_1 i_2$ and $i_1 i_3$ yields the following moduli:

(P138) $\zeta\ \Gamma_{123}(\zeta)$

(P139) $\Gamma_1(\zeta)\ \Gamma_{23}(\zeta)$

(P140) $\Gamma_2(\zeta)\ \Gamma_{13}(\zeta)$

(P141) $\Gamma_3(\zeta)\ \Gamma_{12}(\zeta)$

Remark 5.1
Each modulus listed above is a quadratic form invariant under the corresponding conjugation pair and naturally belongs to the indicated bicomplex subspace. Together, these families provide a complete and systematic classification of all moduli arising from bicomplex structures within the tricomplex algebra $\mathbb{C}_3$.

6. CONJUGATION OF IDEMPOTENT ELEMENTS IN MULTICOMPLEX SPACE $\mathbb{C}_n$

Let

$$\mathbb{C}_n = \mathbb{C}(i_1, i_2, \dots, i_{n-1}, i_n)$$

be the multicomplex space generated by the commuting imaginary units i_k, satisfying

$$i_k^2 = -1, i_k i_l = i_l i_k \ \ (k, l = 1,2, \dots, n).$$

For each imaginary unit i_k , the i_k -conjugation is the linear involution

$$\Gamma_k: \mathbb{C}_n \to \mathbb{C}_n, \Gamma_k(\zeta) = \overline{(\zeta)}_{i_k} ,$$

defined by

$$\Gamma_k(i_k) = -i_k, \Gamma_k(i_j) = i_j \ (j \neq k), \Gamma_k(\zeta_1 + \zeta_2) = \Gamma_k(\zeta_1) + \Gamma_k(\zeta_2), \Gamma_k(\zeta_1 \zeta_2) = \Gamma_k(\zeta_1)\Gamma_k(\zeta_2).$$

Each conjugation is an involutive automorphism:

$$\Gamma_k^2 = \mathrm{id}, \Gamma_k(\zeta\eta) = \Gamma_k(\zeta)\Gamma_k(\eta).$$

6.1. Behavior of Conjugation on Idempotent Elements

Let $e \in \mathbb{C}_n$ be an idempotent element:

$$e^2 = e.$$

Then for any conjugation Γ_k,

$$\left(\Gamma_k(e)\right)^2 = \Gamma_k(e)\Gamma_k(e) = \Gamma_k(e^2) = \Gamma_k(e).$$

Hence $\Gamma_k(e)$ is also an idempotent.

Therefore, conjugation maps idempotent elements to idempotent elements.

6.2. Conjugation and Primitive Idempotents

Let $E \subset \mathbb{C}_n$ denote the set of all idempotents and $P \subset E$ the set of primitive idempotents (minimal non-zero idempotents).

Because each $\mathbb{C}_n$ is an algebra automorphism:

Result.

If e is primitive idempotent idempotent, then

$$\Gamma_k(e) \in P.$$

Thus, conjugation operators act as permutations on the set of primitive idempotents.

Illustrative Examples:

In bicomplex space $\mathbb{C}_2$ there are two idempotents

$$e_{12} = \frac{1 + i_1 i_2}{2}, e_{-12} = \frac{1 - i_1 i_2}{2}.$$

Under conjugation, they satisfy

$$\Gamma_1(e_{12}) = \Gamma_2(e_{12}) = e_{-12}, \quad \Gamma_{12}(e_{12}) = e_{12}$$
$$\Gamma_1(e_{-12}) = \Gamma_2(e_{-12}) = e_{12}, \ \Gamma_{12}(e_{-12}) = e_{-12}$$

In tricomplex space $\mathbb{C}_3$ there are four primitive idempotents

$$c_1 = e_{12}e_{13} = \frac{1}{4}(1 + i_1i_2 + i_1i_3 - i_2i_3), \quad c_2 = e_{12}e_{-13} = \frac{1}{4}(1 + i_1i_2 - i_1i_3 + i_2i_3),$$
$$c_3 = e_{-12}e_{13} = \frac{1}{4}(1 - i_1i_2 + i_1i_3 + i_2i_3), \quad c_4 = e_{-12}e_{-13} = \frac{1}{4}(1 - i_1i_2 - i_1i_3 - i_2i_3).$$

Under all seven conjugations, these primitive idempotents are permuted among themselves. For example:

$$\Gamma_1(c_1) = c_4, \Gamma_2(c_1) = c_3, \Gamma_3(c_1) = c_2, \Gamma_{12}(c_1) = c_2, \Gamma_{13}(c_1) = c_3, \Gamma_{23}(c_1) = c_4, \Gamma_{123}(c_1) = c_1,$$
$$\Gamma_1(c_2) = c_3, \Gamma_2(c_2) = c_4, \Gamma_3(c_2) = c_1, \Gamma_{12}(c_2) = c_1, \Gamma_{13}(c_2) = c_4, \Gamma_{23}(c_2) = c_3, \Gamma_{123}(c_2) = c_2,$$
$$\Gamma_1(c_3) = c_2, \Gamma_2(c_3) = c_1, \Gamma_3(c_3) = c_4, \Gamma_{12}(c_3) = c_4, \Gamma_{13}(c_3) = c_1, \Gamma_{23}(c_3) = c_2, \Gamma_{123}(c_3) = c_3,$$
$$\Gamma_1(c_4) = c_1, \Gamma_2(c_4) = c_2, \Gamma_3(c_4) = c_3, \Gamma_{12}(c_4) = c_3, \Gamma_{13}(c_4) = c_2, \Gamma_{23}(c_4) = c_1, \Gamma_{123}(c_4) = c_4 .$$

Each conjugation thus acts as a permutation on the set $\{c_1, c_2, c_3, c_4\}$, highlighting the symmetry structure inherent in the tricomplex algebra.

CONCLUSION

This study examined the algebraic structure and symmetry of tricomplex numbers through its seven involutive conjugations. We defined these conjugations and derived their fundamental properties under addition, subtraction, and multiplicative relations. By systematically comparing pairs of conjugations, we established a comprehensive set of identities that classify elements of $\mathbb{C}_3$ in terms of invariant subalgebras and twisted modules. A total of 141 identities were obtained through analysis of equality, negation, addition, subtraction, and multiplicative relations among conjugation operators, including 112 canonical conjugation identities. These identities correspond to idempotent projection operators that decompose any tricomplex number into invariant and anti-invariant components, yielding a full canonical decomposition into symmetry-adapted subspaces. We also examined the action of conjugations on primitive idempotent elements. Together, these results extend classical conjugation theory to higher-order multicomplex systems and provide a unified framework for structural analysis, subalgebra classification, and functional formulations.

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